

Violent relaxation in hierarchical clustering

By SIMON D. M. WHITE¹

¹Max-Planck Institut für Astrophysik, Karl-Schwarzschild-Strasse 1, 85740 Garching bei München, Germany

The term “violent relaxation” was coined by Donald Lynden-Bell as a memorable oxymoron describing how a stellar dynamical system relaxes from a chaotic initial state to a quasi-equilibrium. His analysis showed that this process is rapid, even for systems with many stars, and that it leads to equilibria which may plausibly be related to bounded isothermal spheres. I review how numerical simulations have improved our understanding of violent relaxation over the last thirty years. It is clear that the process leads to equilibria which depend strongly on the initial state, but which nevertheless have certain common features. A particularly interesting case concerns objects formed in an expanding universe through dissipationless hierarchical clustering from gaussian initial conditions; these may correspond to galaxy clusters or to the dark halos of galaxies. While such objects display a wide range of shapes and spins, the distributions of these properties depend only weakly on the cosmological context and on the initial spectrum of density fluctuations. Halo density profiles appear to have a universal form with a singular central structure and a characteristic density which depends only on formation epoch. Low mass halos typically have earlier formation times and thus higher characteristic densities than high mass halos.

1. Introduction

By the mid-1960’s it was known that the luminosity profiles of elliptical galaxies are smooth, symmetric, and well fitted by theoretical models based on modifications of the isothermal sphere (see, for example, King 1966). This was perceived as paradoxical since Chandrasekhar’s (1942) calculation of the effects of “collisional” relaxation showed the relevant timescales for elliptical galaxies to be much longer than the age of the universe. Furthermore such two-body relaxation should lead to equipartition of energy and so to radial segregation of stars by mass; this would produce unacceptably large colour gradients in an elliptical galaxy. Thus a relaxation process with a short timescale and no dependence on stellar mass is required to explain the observations.

Although Donald Lynden-Bell was not the first to realise that rapid changes in gravitational potential during protogalactic collapse could provide such a process, his 1967 paper gave a clear description of the mechanisms involved, calculated the relevant timescale, showed how evolution might lead to a state resembling an isothermal sphere but with no mass segregation, gave a first discussion of the reasons why real galaxies would be unable to reach the “most probable” state, and above all gave the process a name that everyone can remember. This paper made strong claims in the inimitable Lynden-Bell style, all of them stimulating, but not all, perhaps, fully justified. After deriving a fourth kind of equilibrium statistics to set beside those of Maxwell-Boltzmann, Einstein-Bose, and Fermi-Dirac, and arguing that incomplete relaxation produces rotationally flattened objects resembling real galaxies, the text concludes by throwing doubt on its own major assumptions and careering off into a discussion of the gravothermal catastrophe, the heat death of the universe, and the origin of the Seyfert phenomenon!

The forthright and provocative style of Lynden-Bell (1967) is undoubtedly responsible both for its success and for its rather mixed reputation among professional dynamicists. The latter can be inferred from the slightly hostile (and in my opinion misguided) reworking in Shu (1978), and from the absence of any discussion of the statistics of violent

relaxation either in the standard textbook of Binney & Tremaine (1987) or in Tremaine, Hénon & Lynden-Bell’s (1986) treatment of mixing processes during violent relaxation. There can be no doubt, however, that the 1967 paper did much to shape our current understanding of how stellar dynamical systems come to equilibrium, and even today it remains both frequently cited and controversial. (See Earn’s contribution to this volume for some illuminating statistics).

Numerical experiments in the 1970’s showed that violent relaxation from realistic initial conditions does *not* lead to a universal final structure, even after accounting for the restriction already discussed explicitly by Lynden-Bell (1967), namely that the finite mass of any real system implies that the most probable statistical distribution can only be realised for orbits with periods shorter than the collapse time of the system, and so for energies more negative than some truncation threshold. The discrepancy was first clearly demonstrated by the collapse simulations of White (1976) and Aarseth & Binney (1978). These showed that the violent collapse of stellar dynamical systems can produce strongly aspherical equilibria, whereas Lynden-Bell’s arguments imply a final state which is spherical, at least in the absence of significant rotation. In numerical experiments the final state clearly remembers nonspherical aspects of the collapse and relaxation phases. Unfortunately the corresponding constraints are difficult to formulate and have so far been ignored in attempts to go beyond the Lynden-Bell’s original analysis and to obtain a deeper understanding of the observed uniformity of elliptical galaxies (*e.g.* Stiavelli & Bertin 1987; White & Narayan 1987; Tremaine 1987). The only clear area of progress since 1967 is in understanding the light profiles at large radii. A number of authors have shown that asymptotically a relation $S \propto r_p^{-3}$ should hold between surface brightness S and projected radius r_p (Aguilar & White 1986; Jaffe 1987; Tremaine 1987); this results from the expected continuity in the distribution of stellar binding energies across the value corresponding to escape from the system.

In the current contribution I will discuss recent work on violent relaxation in a different context, that of the formation of dark halos by hierarchical clustering of dissipationless dark matter in an expanding universe. Most past studies of this process have concentrated on differences in the predicted halo structure as a function of the density of the Universe, and of the nature of the initial fluctuations from which structure forms. Here I will instead highlight certain regularities that emerge from the available numerical data, and argue that in this context violent relaxation actually produces a “universal” density structure which is independent of halo mass, of cosmological parameters, and of the initial fluctuation spectrum. Unfortunately, we do not yet have a physical understanding of this universal structure of the kind which Lynden-Bell (1967) attempted to provide.

2. Hierarchical clustering

An idealised model for the growth of structure in the universe makes the following assumptions.

- (i) The dominant mass component in the universe is the dark matter. This consists of particles which interact only through gravity and whose individual masses are so small that pairwise encounters can be neglected when studying evolution on galactic and supergalactic scales. Dark matter candidates which satisfy this assumption include massive neutrinos, axions, the lightest supersymmetric particle, “jupiters”, and black holes of masses $100M_\odot$ or less. The distribution of dark matter can then be described by its one-particle phase-space distribution $f(x, v, t)$ which evolves according to the collisionless Boltzmann equation, $Df/Dt = 0$.
- (ii) The evolution of the non-dominant component (which we can actually see!) does not

significantly affect that of the dark matter distribution. This assumption clearly breaks down in the inner regions of galaxies but may be acceptable on larger scales.

(iii) At early times the dark matter particles have a nearly uniform spatial distribution and have small velocities relative to the mean expansion of the Universe. The latter requirement eliminates light neutrinos with masses ~ 30 eV since their velocities would not be negligible on the relevant scales. The other candidates mentioned above are still all acceptable.

(iv) If the dark matter density fluctuation field at early times is represented as a superposition of plane waves, $\delta(x) = \int d^3k \delta_k \exp(ik \cdot x)$, then the phases of different waves are independently and randomly distributed on $(0, 2\pi]$. The statistical properties of the random field $\delta(x)$ are then fully represented by its power spectrum $P(k) \propto |\delta_k|^2$, and the distribution of $\delta(x)$ at any arbitrary set of positions (x_1, x_2, \dots) is a multivariate gaussian. Such fields are known as gaussian random fields and are predicted by a wide class of theories for the origin of structure in the universe.

(v) The quantity $k^3 P(k)$ increases with wavenumber k . This is the condition that small objects form first and then aggregate into larger systems as structure grows. All currently popular theories for structure formation satisfy this requirement on the scales relevant to galaxies and galaxy clusters.

Over the last twenty years cosmological N-body simulations have been used extensively to study nonlinear evolution from initial conditions obeying these assumptions. This kind of evolution is generically referred to as “hierarchical clustering”. Here I ask whether evolution from such initial conditions leads to structural regularities in the highly nonlinear, quasi-equilibrium objects which it produces. In concrete terms I ask whether dark halos formed by hierarchical clustering have a universal internal structure independent of their mass, of the density of the Universe, and of the initial power spectrum. Such universality could clearly be ascribed to the effects of violent relaxation as envisaged by Lynden-Bell (1967). Although cosmological simulations have usually been used to compare predictions for large-scale structure with the observed clustering of galaxies, a number of studies have also specifically investigated halo structure. The most useful references in this context are Efstathiou et al. (1988), Frenk et al. (1988), Zurek et al. (1988), Dubinski & Carlberg (1991), Warren et al. (1992), Crone et al. (1994), Navarro et al. (1995, 1996) and Cole & Lacey (1996). An important additional reference is the analytic work of Hoffman & Shaham (1985) and Hoffman (1988) which suggests that the slope of the density profile of a dark halo in its outer regions should depend systematically on the power spectrum of initial density fluctuations and on the density of the universe.

3. Shapes and spins of halos

Since the very earliest simulations it has been clear that hierarchical clustering does not produce near-spherical halos. Rather the equidensity surfaces of many simulated halos can be well fit by a set of concentric ellipsoids with axial ratios differing substantially from unity. Major deviations from this structure can usually be ascribed to non-equilibrium effects such as recent mergers. Halos can be oblate, prolate, or any intermediate shape, and their axial ratios can vary substantially with radius. In the published literature there is some disagreement about the details both of these variations and of the distribution of axis ratios. It seems likely that this reflects, at least in part, the different definitions and algorithms used in computing axial ratios. In fact, a comparison of the results of Frenk et al. (1988), of Dubinski & Carlberg (1991), of Warren et al. (1992) and of Cole

& Lacey (1996) shows that the distribution of halo shapes is similar in all cases. Very few objects are nearly spherical, and roughly half have ratios of longest to shortest axis exceeding two. The intermediate axis length scatters rather uniformly between the other two with a weak preference for near-prolate halos. Although some systematic differences are found as a function of halo mass and of the initial power spectrum, these trends are quite weak.

The angular momentum distribution for dark halos was one of the first of their properties to be systematically studied by numerical simulation (Efstathiou & Jones 1979). Almost all later work has followed this original paper and has characterised the rotation rate of a dark halo by its spin parameter, $\lambda = J|E|^{1/2}/GM^{5/2}$ where J , E and M are respectively the total angular momentum, energy and mass (Barnes & Efstathiou 1987, Efstathiou et al. 1988, Warren et al. 1992, Cole & Lacey 1996). There is general agreement that the distribution of λ is very broad with a median between 0.04 and 0.06 and a range of about a factor of 2.5 between the 20% and 80% points. The median declines very weakly with increasing halo mass and is about 10% of the value expected for a fully rotationally supported system such as a self-gravitating disk. Within a dark halo the rotation velocity varies little with radius and there is a relatively poor correlation between the rotation axes at different radii. This is strongly at variance with the solid body rotation predicted by Lynden-Bell (1967), although Donald was careful to point out that his prediction could not apply far from the centre of a relaxed system. There is at most a very weak dependence of the λ -distribution on $P(k)$ or on Ω , with different authors finding slightly different results. These differences probably reflect different (arbitrary) definitions for the boundary of a dark halo.

Violent relaxation during hierarchical clustering clearly does not produce objects with a universal shape or spin. It is, however, quite striking that the *distributions* of these quantities do seem to be (almost) universal. Study of the formation of individual objects suggests that both their shape and their spin depend on the details of their nonlinear collapse and on their immediate environment at that time. For example, the objects with the largest spin tend to be those which formed by merging of two progenitors of comparable mass, while extreme prolate systems are often related to collapse along filaments. The universality of the spin and shape distributions thus reflects some kind of statistical universality for nonlinear formation paths in hierarchical clustering. Quite different distributions of shapes and spins can be found in models which depart fundamentally from the hierarchical clustering paradigm (*e.g.* White & Ostriker 1990).

4. Density profiles

While the near-universality of the spin and shape distributions of dark halos has always been generally accepted, discussions of halo density profiles have tended to focus on the differences expected in different cosmological contexts. This may stem from two influential papers by Hoffman & Shaham (1985) and Hoffman (1988). These used a spherical infall model to study the nonlinear evolution of structure around a peak of an initially gaussian overdensity field. The first paper showed that if the initial fluctuation power spectrum is a power law, $P(k) \propto k^n$, and if the universe is Einstein-de Sitter, then halos might be expected to show power law density profiles, $\rho \propto r^{-\alpha}$ with $\alpha = (9 + 3n)/(4 + n)$ for $n \geq -1$ and $\alpha = -2$ for more negative n . The second paper showed that steeper density profiles (larger effective values of α) are expected in universes with lower values of Ω . Halo formation is, of course, very far from spherically symmetric in hierarchical clustering, so it was quite surprising when a variety of simulation studies confirmed the predicted trends and even found reasonable quantitative agreement when

α is estimated at radii where the density is a few hundred to a few thousand times the critical value (Quinn et al. 1986, Efstathiou et al. 1988, Zurek et al. 1988, Crone et al. 1994). None of this work comments on any possible dependence of density profile on halo mass, perhaps because the theoretical argument can easily be construed as suggesting that no such dependence should be present.

Although the above studies all conclude that halo density profiles reflect both $P(k)$ and Ω and so provide a possible route to estimating these quantities, I will argue here that they can also be considered a universal outcome of violent relaxation during hierarchical clustering, and to be independent of the larger cosmological context in exactly the same way as the spin and shape distributions. I will also argue that they have a substantial dependence on halo mass which is at least as strong as the dependence on $P(k)$ and Ω .

In a high resolution study of the formation of rich clusters of galaxies in the standard CDM cosmogony Navarro et al (1995) noticed that the density profiles of the dark matter distribution in their simulated clusters could be well described by a simple fitting formula:

$$\frac{\rho(r)}{\rho_{crit}} = \frac{\delta_c}{(r/r_s)(1 + r/r_s)^2}, \quad (4.1)$$

where ρ_{crit} is the critical density for closure, r_s is a characteristic linear scale, and δ_c is a characteristic overdensity. This profile is very similar to the analytic form which Hernquist (1990) proposed as a convenient fit to the light distribution in elliptical galaxies. Both formulae are singular for small r with $\rho \propto r^{-1}$; however equation (4.1) gives $\rho \propto r^{-3}$ at large radii whereas the Hernquist law gives $\rho \propto r^{-4}$. Note that Navarro et al apply their formula only within the conventional “virial radius” r_{200} of a cluster. This is defined to be the radius of the sphere within which the mean density is 200 times the critical value, and it is a good estimate of the region within which a cluster can be assumed to be approximately in hydrostatic equilibrium (see Cole & Lacey 1996).

In follow-up work Navarro et al (1996) found that equation (4.1) is actually a good representation of the density structure of halos of any mass in the standard CDM cosmogony, and that the characteristic overdensity is strongly correlated with halo mass; low mass halos have systematically larger values of δ_c than high mass halos. I illustrate this in Figure 1 which is taken from their paper. Rather than showing the density profiles directly, this plot gives circular velocity profiles for two halos. These are defined through

$$V_c(r) = (GM(r)/r)^{1/2}, \quad (4.2)$$

where $M(r)$ is the total mass contained in a sphere of radius r . A characteristic mass and circular velocity for each halo can then be defined through $M_{200} = M(r_{200})$ and $V_{200} = V_c(r_{200})$. In Figure 1 the radius and the circular velocity are scaled to r_{200} and V_{200} respectively. The two halos shown have M_{200} values of $3 \times 10^{15} M_\odot$ and $3 \times 10^{11} M_\odot$ and so correspond to a rich cluster and to a small galaxy halo. The fact that the two scaled curves do not coincide shows that the density profiles of the two objects differ by more than a simple linear scaling. The small halo corresponds to the curve which rises highest and so has a higher concentration and a larger characteristic overdensity than the rich cluster. Both curves are well fit by equation (4.1). The rich cluster curve could also be fit quite well by the Hernquist formula, but it is clear from Figure 1 that this formula gives a substantially poorer fit to the circular velocity curve of the small halo.

An obvious question is whether the good fit to equation (4.1) is specific to CDM halos or reflects a more general property of hierarchical clustering. In a beautiful recent paper Cole & Lacey (1996) show that this same fitting formula is a very good representation of the mean shape of halo density profiles in simulations of “scale-free” hierarchical clustering in universes with $\Omega = 1$ and $P(k) \propto k^n$. (They consider n values in the range

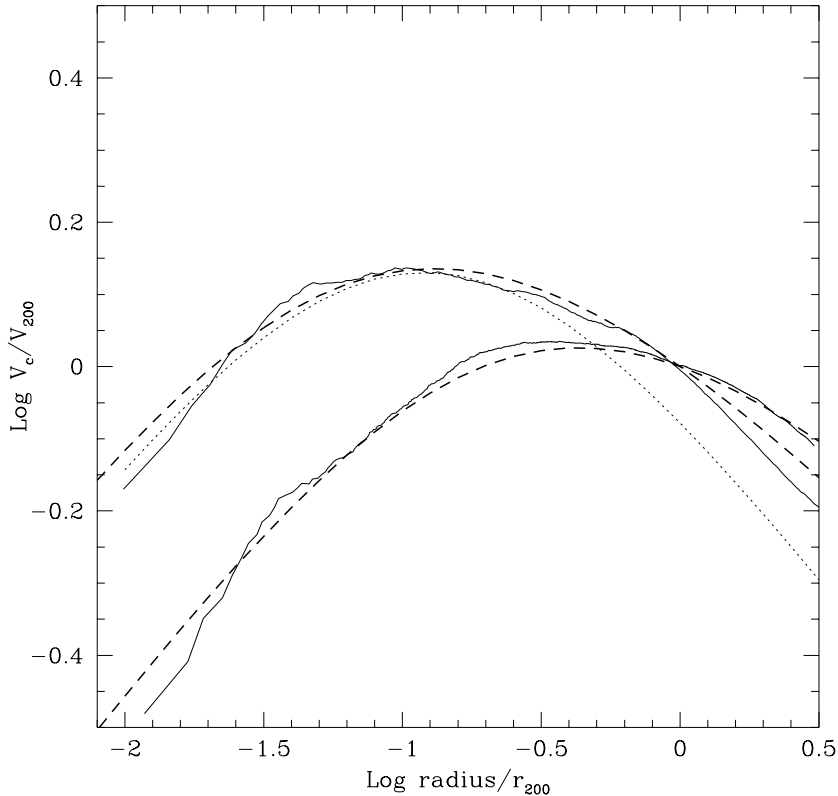


FIGURE 1. Scaled circular velocity curves for two simulated dark halos in a standard CDM universe (solid curves). Velocities and radii are normalised by r_{200} and V_{200} respectively, where (r_{200}, V_{200}) is (172kpc, 86 km/s) for the upper curve and (3.7Mpc, 1860km/s) for the lower one. The dashed lines are the best fits of equation (4.1) to the numerical data, while the dotted line is the best fit of a Hernquist profile to the upper curve in the range $r < r_{200}$.

$-2 \leq n \leq 0$.) In such universes the clustering patterns at different times are statistically equivalent once they are scaled in mass by the characteristic value $M_*(t) \propto t^{4/(3+n)}$. Cole & Lacey demonstrate that for given n the characteristic overdensity of dark halos decreases with increasing M/M_* , just as in the CDM case. Moreover they find that at fixed M/M_* halo profiles have systematically smaller δ_c in universes with more negative n . It seems therefore that equation (4.1) is a good representation of the density profiles of dark halos for dissipationless hierarchical clustering in any Einstein-de Sitter universe.

Our own follow-up work (Navarro, Frenk & White, in preparation) has confirmed Cole & Lacey's results for halos in scale-free universes and has extended the ranges both of halo mass and of radius within a halo for which equation (4.1) is demonstrated to be a good fit; at present these are roughly $0.01M_* \leq M_{200} \leq 100M_*$ and $0.01r_{200} \leq r \leq r_{200}$. We have also tested our fitting formula in low density universes (with and without a cosmological constant) with both power law and CDM-like initial power spectra. In all cases we have tried so far (which cover $0.1 \leq \Omega \leq 1$) the fit remains quite good. At given M/M_* halos have higher overdensities in universes with smaller values of Ω ; at given M/M_* and Ω they have higher overdensities in open universes than in flat universes with a cosmological constant.

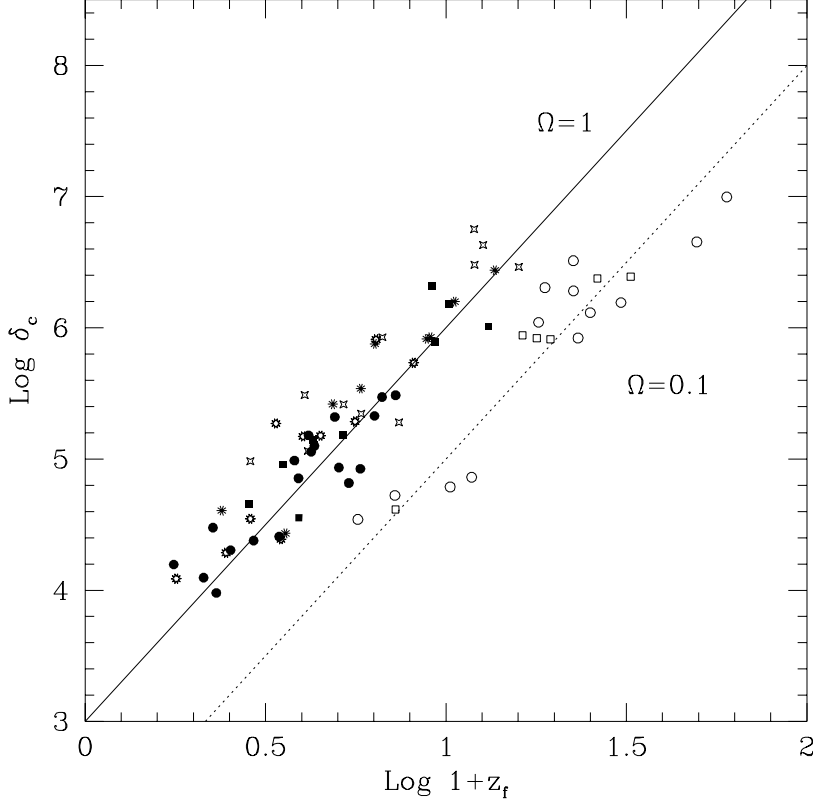


FIGURE 2. Characteristic overdensity is plotted against formation redshift for simulated halos. Each symbol corresponds to a different cosmogony as follows: open squares and open circles are halos in an open universe, $\Omega_0 = 0.1$, with scale-free initial fluctuation spectra with $n = 0$ and $n = -1$ respectively; filled circles are halos in a standard CDM cosmogony; other symbols refer to halos in scale-free Einstein-de Sitter universes with $n = 0, -0.5, -1$ and -1.5 . For each cosmogony halo mass decreases systematically with increasing δ_c and z_f . The two straight lines correspond to equation (4.3) for $\Omega = 1$ and $\Omega = 0.1$.

It turns out that all these systematic dependences of halo concentration on mass, on initial power spectrum, and on cosmological density can be understood quite simply in terms of halo formation times. In hierarchical clustering the high mass objects identified at any given time typically formed much more recently than low mass objects identified at the same time (see, for example, Lacey & Cole 1993, 1994). Similarly, for a given initial power spectrum $P(k)$ and nonlinear mass scale M_* , halos of some fixed M/M_* form at higher redshift in a low density universe than in an Einstein-de Sitter universe. Finally, in scale-free hierarchical clustering the growth of structure is faster (and so typical objects form more recently) in universes with more negative values of the initial power spectrum index n . With a suitable definition of formation redshift z_f all our data are consistent with the very simple relation,

$$\delta_c = 1000\Omega_0(1 + z_f)^3. \quad (4.3)$$

This is shown in Figure 2 which was compiled by Julio Navarro using all the data presently available from our programme of simulations. It is quite remarkable how well the value

of δ_c scales with $(1 + z_f)^3$ as measured directly in the simulations. (For this plot z_f is defined for each halo as the earliest redshift at which its biggest progenitor has more than 10% of its final mass. Other definitions give similar results.)

5. Concluding remarks

The material discussed in the preceding sections suggests that there is indeed a certain universality in the properties of the objects which form by dissipationless hierarchical clustering. Although it is generally agreed that the shapes and spins of dark halos are predicted to have distributions which depend at most weakly on halo mass, on the power spectrum of initial density fluctuations, and on the density of the universe, our claim that the density profiles of objects are also independent of these quantities seems, at first sight, to conflict with earlier work. This discrepancy is, however, only superficial. Previous numerical studies fitted power laws to halo density profiles over a limited range of overdensities, typically $\sim 10^2$ to $\sim 10^4$. As a result, the systematic dependence of formation redshift on power spectrum and Ω , clearly visible from the point distribution in Figure 2, produced systematic variations in the measured slopes. Furthermore, most studies surveyed too small a range of halo masses to notice the systematic mass dependence. Direct comparison of our predictions with the numerical results of earlier studies shows good agreement in almost all cases.

It seems, therefore, that violent relaxation in hierarchical clustering leads to objects with universal density profiles and universal distributions of shape and spin. Gravitational potential fluctuations during the collision and merging processes which characterize hierarchical clustering are evidently strong enough to cause convergent evolution of the kind envisaged by Lynden-Bell (1967). Since Donald's detailed statistical arguments provide at best a partial explanation of this process, further work is needed to achieve a deeper understanding of the simplicity which emerges from numerical experiment.

I would like to thank my collaborators, Julio Navarro and Carlos Frenk, for permission to reproduce results from our joint research programme.

REFERENCES

- AARSETH, S.J. & BINNEY, J. 1978 *MNRAS* **185**, 227.
 AGUILAR, L. & WHITE, S.D.M. 1986 *ApJ* **307**, 97.
 BINNEY, J. & TREMAINE, S.D. 1987 *Galactic Dynamics*, Princeton Univ. Press, Princeton
 BARNES, J. & EFSTATHIOU, G. 1987 *ApJ* **319**, 575.
 CHANDRASEKHAR, S. 1942 *Principles of Stellar Dynamics*, Dover, New York.
 CRONE, M., EVRARD, A.E. & RICHSTONE, D.O. 1994 *ApJ* **434**, 402.
 COLE, S.M. & LACEY, C.G. 1996 *MNRAS*, in press.
 DUBINSKI, J. & CARLBERG, R. 1991 *ApJ* **378**, 496.
 EFSTATHIOU, G. & JONES, B.J.T. 1979 *MNRAS* **186**, 133.
 EFSTATHIOU, G., FRENK, C.S., WHITE, S.D.M. & DAVIS, M. 1988 *MNRAS* **235**, 715.
 FRENK, C.S., WHITE, S.D.M., DAVIS, M. & EFSTATHIOU, G. 1988 *ApJ* **327**, 507.
 HERNQUIST, L. 1990 *ApJ* **356**, 359.
 HOFFMAN, Y. 1988 *ApJ* **328**, 489.
 HOFFMAN, Y. & SHAHAM, J. 1985 *ApJ* **297**, 16.
 JAFFE, W. 1987 *Structure and Dynamics of Elliptical Galaxies* (ed. P.T. de Zeeuw), p 511.
 KING, I.R. 1966 *AJ* **71**, 64.

- LACEY C.G. & COLE, S.M. 1993 *MNRAS* **262**, 627.
- LACEY C.G. & COLE, S.M. 1994 *MNRAS* **271**, 676.
- LYNDEN-BELL D. 1967 *MNRAS* **136**, 101.
- NAVARRO, J.F., FRENK, C.S. & WHITE, S.D.M. 1995 *MNRAS* **275**, 720.
- NAVARRO, J.F., FRENK, C.S. & WHITE, S.D.M. 1996 *ApJ*, in press.
- QUINN, P.J., SALMON, J.K. & ZUREK, W.H. 1986 *Nature* **322**, 329.
- SHU, F.H. 1978 *ApJ* **225**, 83.
- STIAVELLI, M. & BERTIN, G. 1987 *MNRAS* **217**, 735.
- TREMAINE, S.D. 1987 *Structure and Dynamics of Elliptical Galaxies* (ed. P.T. de Zeeuw), p 367.
- TREMAINE, S.D., HÉNON, M. & LYNDEN-BELL D. 1986 *MNRAS* **219**, 285.
- WARREN, M.S., QUINN, P.J., SALMON, J.K. & ZUREK, W.H. 1992 *ApJ* **399**, 405.
- WHITE S.D.M. 1976 *MNRAS* **177**, 717.
- WHITE, S.D.M. & NARAYAN, R. 1987 *MNRAS* **229**, 103.
- WHITE, S.D.M. & OSTRICKER, J.P. 1990 *ApJ* **349**, 22.
- ZUREK, W.H., QUINN, P.J. & SALMON, J.K. 1988 *ApJ* **330**, 519.